Parabolic trajectories and Lie's contact transformations

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But I may illustrate these sorts of problems,
and make familiar the Method of reducinfg them
to Equations;and since Arts are more easily learned
by Examples than Precepts, I have thought fit to
adjoin the solution of the following Problems.

Isaac Newton
Universal Arithmetick

1. Introduction

The great mathematician Sophus Lie invented infinitesimal transformations to solve differential equations that, we know, are the mean, at least from the time of Newton, to study in deph the diverse aspects of the natural world. The idea of Lie was to apply to these equations the theory of groups that Galois has apllied to algebraic equations.
Though Hamilton, in some way, was close to the idea, contact transformations were brought out explicitly by Sophus Lie as early as in 1872 when in a seminal work he pointed out the importance of transformations beginning and epoch change that goes on to the present [10].

2. Characteristic Function

Sophus Lie saw clearly the importance and mutual relationship of three mathematical concepts:
Infinitesimal groups, symmetries of differential equations and geometry, given form to the famous Lie groups that are the basis of new developments and one of the pillars of the modern vision of solution of natural problems.
Lie defined a lineal element as a point, (x,y) and a line through it whose slope is, p. An infinity of lineal elements (for example a curve)forms a union of elements. Any transformation on the coordinates which transforms every union of elements into a union was called by Lie a contact transformation. On the other hand, a transformation of coordinates point to point was called point transformation.
Contact transformations are the core of Lie theory. According to Lie, the infinitesimal contact transformation in a plane converted the lineal element, (x,y,p) into a neigboring lineal element,(x+dx,y+dy,p+dp),following some rules. We introduce,x, and,y, as coordinates of a point and ,p, as the derivative of ,y, with respect of ,x.

Sophus Lie contact transformations are determined completely by a function that he called characterristic function, W. Any infinitesimal contact transformation in three coordinates, x,y and p can be written,

$$Bf = [Wf] - W\frac{\partial f}{\partial y} \tag{1}$$

Where the symbol $[Wf]$ means,

$$[Wf] = \begin{vmatrix} W_p & W_x + pW_y \\ F_p & F_x + pF_y \end{vmatrix} \tag{2}$$

Where sub-indexes represent partial derivatives.

Though we are considering a physical problem the Lie treatment of it is purely geometric. That was the greatness of Lie theory as pointed out by his friend Felix Klein in a famous lecture in 1893 [9].

According to Lie the infinitesimal contact transformation in mechanic problem [2]
have a characteristic function of the type,

$$W = \phi(x, y)\sqrt{1 + p^2} \tag{3}$$

As it was said above, in all cases ,$p = \dfrac{dy}{dx}$

3. Gravitational field

From elementary dynamical theory we know that the path of a particle in a vertical plane to be a parabolic trajectory,

$$y = \frac{a}{2g} - \frac{b^2}{2g} - \frac{g}{2b^2}(x - d)^2 \tag{4}$$

Where a,b and d are constants, and g is the gravity acceleration.
In general the derivative of the trajectory is F=0, being F,

$$F = p + \frac{gx}{b^2} - c \tag{5}$$

The constant c can be written,$c = \dfrac{gd}{b^2}$

Also the dynamical theory shows that,

$$\frac{\sqrt{a-2gy}}{\sqrt{1+p^2}} = b \tag{6}$$

It's easy to see that this is the Euler's equation of the variational problem in the case of a particle in the gravitational field.

4. Gravitational characteristic function

Bearing in mind the above, and following Lie's calculus of infinitesimal contact transformation we can check that the characteristic function is,

$$W = \frac{\sqrt{1+p^2}}{\sqrt{a-2gy}} \tag{7}$$

The main thing with the Lie chacteristic function is that with this simple formula you can get allresults in the field of motion of a particle on a vertical plane so proving the value of Lie's group theory to solve completely the problem.
Taking into account (1) and (2)we can compute the infinitesimal contact transformation
that can be written,

$$Bf = \beta\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \pi\frac{\partial f}{\partial p} \tag{8}$$

$$\beta = \frac{p}{\sqrt{1+p^2}}\frac{1}{\sqrt{a-2gy}} \tag{9}$$

$$\eta = \frac{-1}{\sqrt{1+p^2}}\frac{1}{\sqrt{a-2gy}} \tag{10}$$

$$\pi = \frac{-gp\sqrt{1+p^2}}{\sqrt{(a-2gy)^3}} \tag{11}$$

Bf is the infinitesimal contact transformation of the gravitational field in this

case
So far,we can see that all equations are written on the language of differential geometry, but according to Lie there is a correspondence between Infinitesimal transformations and one-parameter group. In our case this group is,

$$
\begin{aligned}
X &= x + \frac{p}{\sqrt{1+p^2}} \frac{t}{\sqrt{a-2gy}} \\
Y &= y - \frac{p}{\sqrt{1+p^2}} \frac{t}{\sqrt{a-2gy}} \\
P &= p - \frac{gpt\sqrt{1+p^2}}{\sqrt{(a-2gy)}^3}
\end{aligned}
\tag{12}
$$

Where, t, is the parameter.
Applying the infinitesimal transformation B to F (see (5)) yields,

$$
BF = \frac{1+p^2}{(a-2gy)\sqrt{a-2gy}\sqrt{1+p^2}} - \frac{\sqrt{1+p^2}}{\sqrt{(a-2gy)^3}} \tag{13}
$$

We can see that BF=0 because F is the derivative of the trajectory and W is the characteristic function. Conversely, F must be lineal in ,x, and therefore the trajectory must be cuadratic (parabolic).

5. Functions in involution

Another important concept in Lie's theory is the "involution" Two functions are in involution when both are invariants through the same differential equation. To prove it in our case we choose as functions the F of above and the Euler's equation of the variational problem. We will call it G put in th form,

$$
G = b\sqrt{1+p^2} - \sqrt{a-2gy} \tag{14}
$$

then we get,

$$
[FG] = \begin{vmatrix} F_p & F_x + pF_y \\ G_p & G_x + pG_y \end{vmatrix} \tag{15}
$$

F and G are said to be in involution.
That means that the parabolic trajectory which derivative is F,admits the same differential equation That G which is the Euler's equation of the dynamical system.
All results are based on Lie group theory which germ can be tracked to his work of 1872, and even a little earlier (Repräsentation der Imaginären der Plangeometrie,1869,[3])

6. Conclusion

This paper is devoted to Lie infinitesimal c0ntact trasformations, but in order to clarify ideas we applied them to the well-known topic of the parabolic trajectory in a gravitational field, Two main reasons can be found for that:
1. To use all the background of the elementary dynamical theory, including calculus of variations in conservative systems.
2. To prove that all developments are purely geometrics, expressing one of the main characteristic of Lie theory.
As Sophus Lie himself remarked, the theory of tranformation groups gave order and clarity to the dispersion of methods of solution of differential equations (both partial and ordinaries one)[10],[4]and [11].
Finally, we have to bear in mind the relationship between infinitesimal transformations and associated one-parameter group.
If one infinitesimal transformation were repeated infinitely, that would define a one-parameter group [10].

7. References

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